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Rayleigh waves in electrostrictive dielectric solids

G. PARIA

Shri Govindram Seksaria Institute of Technology and Science, Indore (M.P.), India

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SUMMARY
The possibility of the propagation of Rayleigh waves in an electrostrictive dielectric medium is investigated. It is shown that such waves can propagate, but they induce some body forces and surface tractions.

1. Introduction

Though the theory of the effect of electrostriction on dielectric solids is known [1, 2, 3] for some years past, the solution of particular problems based on the theory seems to be rather rare [4, 5]. Recently, Paria [6] has solved some problems including Love waves. In the present paper, the possibility of the propagation of Rayleigh waves in such media is investigated. The mathematical technique used here appeals to the linear theory of elastic solids even though the present problem is a non-linear one. It is shown that Rayleigh waves can propagate, but they induce some body forces and surface tractions.

2. General theory

We consider an electrostrictive dielectric solid whose elastic and electric properties are homogeneous and isotropic when there is no stress or no strain. When deformation is produced, the electric properties may become anisotropic, and in such situations the coefficients of anisotropy depend upon the strain developed within the medium.

In such media, the stress tensor σ_{ij} is related to the strain tensor e_{ij} and the electric field E_i as [3]

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + a_1 E_i E_j + b_1 E_m E_m \delta_{ij} \tag{2.1}$$

where the symbols have their usual meanings. By definition, we have

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.2}$$

In an anisotropic medium, the electric displacement D_i is related to E_i as

$$D_i = k_{ij} E_j \tag{2.3}$$

where the anisotropic coefficients k_{ij} under isothermal conditions are given by

$$k_{ij} = k\delta_{ij} + K_1 e_{ij} + K_2 e_{kk}\delta_{ij} . \tag{2.4}$$

The constants K, K_1 and K_2 are characteristic for the electrostrictive property and are determined experimentally. It may be shown that [3] the constants a_1 and b_1 in (2.1) are expressible in terms of these quantities as

$$a_1 = 2K - K_1 , \quad b_1 = -(K + K_2) . \tag{2.5}$$

The stress equations of motion are

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho X_i = \rho \frac{\partial^2 u_i}{\partial t^2} , \tag{2.6}$$

and Maxwell's electrical equations are

$$\operatorname{curl} \boldsymbol{E} = 0\,, \tag{2.7}$$

$$\operatorname{div} \boldsymbol{D} = 0\,. \tag{2.8}$$

Equations (2.1) to (2.8) are the fundamental equations. They are to be solved under prescribed electrical and mechanical boundary and initial conditions.

3. Rayleigh waves

We consider a semi-infinite medium bounded by the plane $z=0$, and the positive direction of the z-axis is taken into the medium. We consider the possibility of the propagation a surface wave of the Rayleigh type such that the disturbance penetrates only a little into the interior. Let the waves propagate parallel to the x-axis with velocity c. As in the purely elastic problem we assume that the displacement components are

$$\begin{aligned} u_1 &= -i(s\,e^{-az}+Ab\,e^{-bz})\,e^{is(x-ct)}\,, \\ u_2 &= 0\,, \\ u_3 &= (a\,e^{-az}+As\,e^{-bz})\,e^{is(x-ct)}\,, \end{aligned} \tag{3.1}$$

where $i=\sqrt{-1}$, $a>0$, $b>0$, and c is the wave velocity. In (3.1), we consider only the real parts. The coordinates and displacements may be taken as dimensionless.

By (2.2), the strain components are

$$\begin{aligned} e_{11} &= s(s\,e^{-az}+Ab\,e^{-bz})\,e^{is(x-ct)}\,, \\ e_{33} &= -(a^2\,e^{-az}+Asb\,e^{-bz})\,e^{is(x-ct)}\,, \\ e_{13} &= e_{31} = \frac{i}{2}\,[2sa\,e^{-az}+A(b^2+s^2)\,e^{-bz}]\,e^{is(x-ct)}\,, \end{aligned} \tag{3.2}$$

$$e_{22}=0\,,\quad e_{12}=0\,,\quad e_{23}=0\,.$$

The dilatation e_{kk} is given by

$$e_{kk} = (s^2-a^2)\,e^{-az}\,e^{is(x-ct)}\,. \tag{3.3}$$

From (2.4) we get

$$\begin{aligned} k_{11} &= K-\{(b_2 s^2+K_2 a^2)\,e^{-az}-AbsK_1\,e^{-bz}\}\,e^{is(x-ct)}\,, \\ k_{22} &= K+K_2(s^2-a^2)\,e^{-az}\,e^{is(x-ct)}\,, \\ k_{33} &= K+\{(b^2 a^2+K_2 s^2)\,e^{-az}-AbsK_1\,e^{-bz}\}\,e^{is(x-ct)}\,, \\ k_{13} &= k_{31} = \frac{iK_1}{2}\,\{2sa\,e^{-bz}+A(b^2+s^2)\,e^{-bz}\}\,e^{is(x-ct)}\,, \\ k_{12} &= 0\,,\quad k_{23}=0\,, \end{aligned} \tag{3.4}$$

where

$$b_2 = -(K_1+K_2)\,.$$

To satisfy (2.7) we take

$$\boldsymbol{E} = -\operatorname{grad}\phi \tag{3.5}$$

and assume that ϕ is independent of y. Then (2.3) gives

$$\begin{aligned} D_1 &= -k_{11}\phi_x-k_{13}\phi_z\,, \\ D_2 &= 0\,, \\ D_3 &= -k_{31}\phi_x-k_{33}\phi_z \end{aligned} \tag{3.6}$$

where ϕ_x and ϕ_z are the derivatives of ϕ with respect to x and z respectively.

Using (3.4) and (3.6) in (2.8) we get

$$\begin{aligned}K(\phi_{xx}+\phi_{zz})+[&AbsK_1(\phi_{xx}-\phi_{zz})\,\mathrm{e}^{-bz}\\ &-\{(b_2s^2+K_2a^2)\phi_{xx}-(b_2a^2+K_2s^2)\phi_{zz}\}\,\mathrm{e}^{-az}\\ &+iK_1\{2sa\,\mathrm{e}^{-az}+A(b^2+s^2)\mathrm{e}^{-bz}\}\,\phi_{xz}\\ &-i\{sb_2(s^2-a^2)\mathrm{e}^{-az}+\tfrac{1}{2}AK_1b(b^2-s^2)\mathrm{e}^{-bz}\}\,\phi_x\\ &+\{ab_2(s^2-a^2)\mathrm{e}^{-az}+\tfrac{1}{2}AK_1s(b^2-s^2)\mathrm{e}^{-bz}\}\,\phi_z]\\ &\qquad\qquad\times\mathrm{e}^{is(x-ct)}=0\,.\end{aligned}\tag{3.7}$$

Let us assume

$$\phi(x,z,t)=\sum_{n=0}^{\infty}\phi_n(z)\,\mathrm{e}^{ins(x-ct)}\,.\tag{3.8}$$

Inserting this expression for ϕ in (3.7) and equating the different coefficients of $\mathrm{e}^{ins(x-ct)}$ to zero, we get the following system of equations to determine ϕ_n, where ϕ'_n and ϕ''_n are the first and second derivatives respectively of ϕ_n with respect to z.

$$\begin{aligned}&\phi''_0=0\,,\\ &K(\phi''_1-s^2\phi_1)+\{ab_2(s^2-a^2)\mathrm{e}^{-az}+\tfrac{1}{2}AK_1s(b^2-s^2)\mathrm{e}^{-bz}\}\,\phi'_0=0\,.\end{aligned}\tag{3.9}$$

$$\begin{aligned}K[\phi''_{n+1}-(n+1)^2s^2\phi_{n+1}]&-AbsK_1(\phi''_n+n^2s^2\phi_n)\,\mathrm{e}^{-bz}\\ &+\{(b_2s^2+K_2a^2)n^2s^2\phi_n+(b_2a^2+K_2s^2)\phi''_n\}\,\mathrm{e}^{-az}\\ &-K_1\{2sa\,\mathrm{e}^{-az}+A(b^2+s^2)\mathrm{e}^{-bz}\}\,ns\phi'_n\\ &+\{sb_2(s^2-a^2)\mathrm{e}^{-az}+\tfrac{1}{2}AK_1b(b^2-s^2)\mathrm{e}^{-bz}\}\,ns\phi_n\\ &+\{ab_2(s^2-a^2)\mathrm{e}^{-az}+\tfrac{1}{2}AK_1s(b^2-s^2)\mathrm{e}^{-bz}\}\,\phi'_n=0\,,\end{aligned}\tag{3.10}$$

where $n=1, 2, 3, \ldots$.

Let the electric potential be prescribed on the surface $z=0$ as

$$\phi=V\,\mathrm{e}^{is(x-ct)}\tag{3.11}$$

and let $\phi\to 0$ as $z\to\infty$. This implies the conditions

$$\phi_0=0\,,\quad \phi_1=V\,,\quad \phi_2=\phi_3=\ldots=\phi_n=\ldots=0\,,\ \text{at}\ z=0\,,\tag{3.12}$$

and $\phi_i\to 0$ $(i=0, 1, 2, \ldots)$ as $z\to\infty$.

Then, from (3.9) we get

$$\phi_0=0\,,\quad \phi_1=V\,\mathrm{e}^{-sz}\,.\tag{3.13}$$

From (3.10), we obtain

$$\begin{aligned}\phi_2=&-\frac{Vs(s-a)}{K(3s+a)}\{b_2a+(b_2-K_1)s\}\,[\mathrm{e}^{-sz}-\mathrm{e}^{-az}]\,\mathrm{e}^{-sz}\\ &-\frac{K_1}{2K}As(s-b)\,V[\mathrm{e}^{-sz}-\mathrm{e}^{-bz}]\,\mathrm{e}^{-sz}\,.\end{aligned}\tag{3.14}$$

Similarly, the other functions ϕ_3, ϕ_4, ... may be calculated. It is found that $\phi_2\to 0$ as $K_1\to 0$, $K_2\to 0$ (i.e., $b_2\to 0$). Thus, ϕ_2 represents the first order approximation to the effect of the electrostriction on the potential distribution, while ϕ_3, ϕ_4, ... are higher order effects.

Using (3.2), (3.3) and (3.5) in (2.1), we get the stress components as

$$\begin{aligned}\sigma_{11}&=[\lambda(s^2-a^2)\mathrm{e}^{-az}+2\mu s(s\,\mathrm{e}^{-az}+Ab\,\mathrm{e}^{-bz})]\,\mathrm{e}^{is(x-ct)}\\ &\qquad+(a_1+b_1)\,\phi_x^2+b_1\,\phi_z^2\,,\\ \sigma_{22}&=\lambda(s^2-a^2)\mathrm{e}^{-az}\,\mathrm{e}^{is(x-ct)}+b_1(\phi_x^2+\phi_z^2)\,,\\ \sigma_{12}&=0\,,\end{aligned}$$

$$\sigma_{31}=\sigma_{13}=\mu i[2sa\,e^{-az}+A(b^2+s^2)e^{-bz}]\,e^{is(x-ct)}+a_1\phi_x\phi_z\,,$$
$$\sigma_{32}=\sigma_{23}=0\,,$$
$$\sigma_{33}=[\lambda(s^2-a^2)e^{-az}-2\mu(a^2e^{-az}+Asb\,e^{-bz})]\,e^{is(x-ct)}+b_1\phi_x^2+(a_1+b_1)\phi_z^2\,. \tag{3.15}$$

As in the purely elastic problem, let us assume that the surface $z=0$ is free from tractions arising from mechanical displacement. This implies that the coefficients of $\exp\{is(x-ct)\}$ in σ_{31} and σ_{33} in (3.15) are zero when $z=0$. Thus

$$2sa+A(b^2+s^2)=0\,,\quad \lambda(s^2-a^2)-2\mu(a^2+Asb)=0\,. \tag{3.16a}$$

Eliminating A we get

$$(b^2+s^2)\{(\lambda+2\mu)a^2-\lambda s^2\}=4\mu s^2ab\,. \tag{3.16}$$

Using (3.15) and (3.5) in (2.6), it is found that the equation corresponding to u_2 vanishes identically with $X_2=0$, while the other two equations are satisfied if

$$a=s\left(1-\frac{c^2}{c_1^2}\right)^{\frac{1}{2}},\quad b=s\left(1-\frac{c^2}{c_2^2}\right)^{\frac{1}{2}}, \tag{3.17}$$

$$\begin{aligned} X_1&=-\frac{1}{\rho}\,[2(a_1+b_1)\phi_x\phi_{xx}+(a_1+2b_1)\phi_z\phi_{xz}+a_1\phi_x\phi_{zz}]\,,\\ X_3&=-\frac{1}{\rho}\,[a_1\phi_{xx}\phi_z+(a_1+2b_1)\phi_x\phi_{xz}+2(a_1+b_1)\phi_z\phi_{zz}]\,, \end{aligned} \tag{3.18}$$

where

$$c_1^2=\frac{\lambda+2\mu}{\rho},\quad c_2^2=\frac{\mu}{\rho}. \tag{3.19}$$

Since a and b have been assumed to be real and positive, we get from (3.17) and (3.19) the condition

$$c<c_2<c_1\,.$$

Using (3.17) and (3.19) in (3.16), we get the frequency equation for the determination of c as

$$(2-\alpha)^4=4(1-\alpha^2)(1-m\alpha^2) \tag{3.20}$$

where

$$\alpha=\frac{c}{c_2},\quad m=\frac{c_2^2}{c_1^2}. \tag{3.21}$$

Equation (3.20) is identical with the frequency equation for Rayleigh waves in a purely elastic medium.

Thus, it is possible for Rayleigh waves to propagate in an electrostrictive dielectric medium. However, the electric field introduces the body forces (3.18) and the surface tractions $\sigma_{31}(0)$ and $\sigma_{33}(0)$ in (3.15) where only the contributions due to ϕ are to be calculated for this purpose.

4. Approximate calculation of body forces and surface tractions

Using (3.8), (3.13), and (3.14) in (3.18), and keeping only the first powers in K_1 and K_2, we get the induced body forces as

$$X_1=\frac{i}{\rho}\,V^2s^2e^{-2sz}[A_1e^{-az}\{6(a_1+b_1)s^2-(5a_1+6b_1)sa-a_1a^2\}+B_1e^{-bz}\{6(a_1+b_1)s^2-(5a_1+6b_1)sb-a_1b^2\}]\,e^{i3s(x-ct)}\,,$$

$$X_3 = \frac{1}{\rho} V^2 s^2 e^{-2sz} [A_1 e^{-az} \{(5a_1+4b_1)s^2 - (3a_1+2b_1)sa - 2(a_1+b_1)a^2\} + B_1 e^{-bz} \{(5a_1+4b_1)s^2 - (3a_1+2b_1)sb - 2(a_1+b_1)b^2\}] e^{i3s(x-ct)}, \tag{4.1}$$

where

$$A_1 = \frac{(s-a)}{K(3s+a)} \{b_2 a + (b_2 - K_1)s\}, \quad B_1 = \frac{K_1 A(s-b)}{2K}, \tag{4.2}$$

while A, a and b are given by (3.16a) and (3.17). Evidently $A_1 \to 0$, $B_1 \to 0$ as $K_1 \to 0$, $K_2 \to 0$.

From (4.1), it is seen that $X_1 \to 0$, $X_3 \to 0$ as $K_1 \to 0$, $K_2 \to 0$. This means that the induced body forces are first order effects of the electrostriction. If the dielectric is not electrostrictive, these body forces do not appear.

From (3.15) we get the surface tractions at $z=0$ as

$$\begin{aligned} (\sigma_{31})_{z=0} &= is^2 V^2 a_1 [e^{2is(x-ct)} + \{A_1(s-a) + B_1(s-b)\} e^{3is(x-ct)}, \\ (\sigma_{33})_{z=0} &= s^2 V^2 [a_1 e^{2is(x-ct)} - 2(a_1+b_1)\{A_1(s-a)+B_1(s-b)\} e^{3is(x-ct)}]. \end{aligned} \tag{4.3}$$

These surface tractions are induced by the potential amplitude V, and vanish when $V=0$. But they do not vanish when $A_1=0$, $B_1=0$. This shows that they are present even when the dielectric material is not electrostrictive.

5. Conclusion

It has been shown that an electrostrictive dielectric material can propagate Rayleigh waves. The applied surface potential induces body forces and surface tractions. The body forces are not induced if the dielectric has not the property of electrostriction. The surface tractions are however present even when the electrostrictive property is ignored.

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